Syllabus for the problem solving class

I. Problem Solving Techniques (heuristics)

a) search for a pattern
b) draw a figure/diagram
c) formulate an equivalent problem
d) modify a problem
e) divide into cases
f) work backwards
g) argue by contradiction
h) exploit symmetry (geometric and algebraic)
i) pursue parity
j) consider extreme cases
k) generalize
l) look for invariants

II. Principles

a) Induction and Strong Induction
b) Recursion
c) Extreme Principle
d) Pigeonhole principle
e) Inclusion-Exclusion Principle

III. Additional topics:

a) graph theory
b) generating functions
c) complex numbers
d) geometry
e) inequalities
f) combinatorics
g) number theory
h) coloring problems
i) polynomials
j) sequences and series
k) games

Resources:
Paul Zeitz, ***The Art and Craft of Problem Solving***
Arthur Engel, ***Problem-Solving Strategies***
Loren C. Larson, ***Problem-Solving Through Problems***
George Polya, ***Mathematical Discovery***

I.

A. Search for a pattern problems

0. Define $f(x) = \dfrac{1}{(1-x)}$ and denote r iterations ($f \circ f .... \circ f(x)$ r times) of the function f by f^r. Compute $f^{1999}(2000)$.

1. Justify that a set of n distinct elements has exactly 2^n distinct subsets.
2. Let x_1, x_2, x_3... be a sequence of nonzero real numbers satisfying $x_n = \dfrac{x_{n-2} \cdot x_{n-1}}{2x_{n-2} - x_{n-1}}$, for n=3, 4, 5...

Establish necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n.

3. Find positive numbers n and $a_1, a_2, ..., a_n$, such that $a_1 + a_2 + ... + a_n = 1000$ and the product $a_1 \cdot a_2 \cdot .... \cdot a_n$ is as large as possible.

4. Determine the number of odd binomial coefficients in the expansion of $(x+y)^{1000}$.

5. A great circle is a circle drawn on a sphere that is an "equator,"i.e., its center is also the center of the sphere. There are n great circle on a sphere, no three of which meet at any point. Find the number of regions n great circles divide the sphere.

6. A variation of the Josephus Problem: a group of n people are standing in a circle, numbered consecutively clockwise from 1 to n. Starting with a person #2, we remove every other person, proceeding clockwise. For example, if n=6, the people are removed in the order 2, 4, 6, 3, 1, and the last person remaining is #5. Let j(n) denote the last person remaining.
a) Compute j(n) for n=2, 3, ...25
b) Find a way to compute j(n) for any positive integer n>1. You may not get a "nice" formula, but try to find a convenient algorithm that is easy to compute by hand or a machine.

B. **Draw a figure**

1. A chord of constant length slides around a semicircle. The midpoint of the chord and the projections of its ends upon the base form the vertices of a triangle. Prove that the triangle is isosceles and never changes its shape.

2. If a and b are positive integers with no common factor , show that:

$$\left[\frac{a}{b}\right]+\left[\frac{2a}{b}\right]+\left[\frac{3a}{b}\right]+\ldots.+\left[\frac{(b-1)a}{b}\right]=\frac{(a-1)(b-1)}{2},$$

where $[\ ]$ represents the greatest integer function.

3. Mr. and Mrs. Wenstein recently attended a party at which there were three other couples. Various handshakes took place. No one shook hands with her/his spouse, no one shook hands with the same person twice, and of course, no one shook her/his own hand. At the end of the party, Mr. Weinstein asked each person, including his wife, how many hands did she or she had shaken. To his surprise each gave a different answer. How many hands did Mrs. Weinstein shake?

C. **Formulate an equivalent problem**

1. Solve: $x^4+x^3+x^2+x+1=0$.
2. P is a point inside a given triangle ABC, D, E, F, are the feet of perpendiculars from P to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, respectively. Find all P for which $\frac{BC}{PD}+\frac{CA}{PE}+\frac{AB}{PF}$ is minimal. (intro to inequalities).
3.

D. **Modify the problem**

1. Given positive numbers, a, b, c, d, prove:

$$\frac{a^3+b^3+c^3}{a+b+c}+\frac{b^3+c^3+d^3}{b+c+d}+\frac{c^3+d^3+a^3}{c+d+a}+\frac{d^3+a^3+b^3}{d+a+b}\geq a^2+b^2+c^2+d^2$$

2. Let C be any point on the line segment AB, between A and B, and let semicircles be drawn on the same side of AB with AB, AC, and CB as diameters. Also let D be a point on the semicircle having diameter AB such that CD is perpendicular to AB, and let E and F be points on the semicircles having diameters AC and CB, respectively, such that EF is a segment of their common tangent. Show that ECFD is a rectangle.

3. Prove that there do not exist positive integers x, y, z, such that: $x^2+y^2+z^2=2xyz$.
4. Evaluate: $\int_0^\infty e^{-x^2}dx$.

(int)

E. **Divide into cases**

1. Prove that the measure of an inscribed in a circle angle equals one-half the central angle which subtends the same arc.

2. Prove that the area of a lattice triangle equals to $A_{lattice\ \ polygon} = I + \frac{B}{2} - 1$.

3. Determine F(x), if for all real x and y: $F(x)F(y) - F(xy) = x + y$.

F. **Work backwards**

1. Let α be a fixed real number, $0 < \alpha < \pi$, and let $F(\theta) = \dfrac{\sin\theta + \sin(\theta + \alpha)}{\cos\theta - \cos(\theta + \alpha)}$, where $0 \le \theta \le \pi - \theta$, show that $F(\theta)$ is a constant.

2. Let a, b, c denote the lengths of the sides of a triangle, show that

$$3(ab + bc + ca) \le (a + b + c)^2 \le 4(ab + bc + ca)$$

G. **Agrue by contradiction**

1. Given that a, b, c are odd integers, prove that $ax^2 + bx + c = 0$ cannot have a rational root.

H. **Exploit symmetry**

1. Equilateral triangles ABK, BCL, CDM, DAN are constructed inside the square ABCD. Prove that the midpoints of the four segments KL, LM, .MN, NK and the midpoints of eight segments AK, BK, BL, CL, CM, DM, AN are the twelve vertices of a regular dodecagon.

2. Determine all values of x which satisfy: $\tan x = \tan(x + 10^\circ)\tan(x + 20^\circ)\tan(x + 30^\circ)$.

I. **Pursue parity**

1. Let there be given nine lattice points in three-dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points.

2. a) Is it possible to trace a path along the arcs of figure I which traverses each arc once and only once?

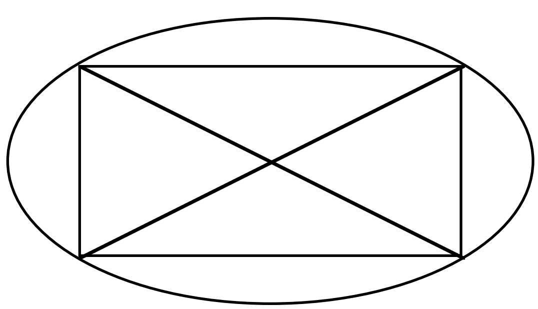

Figure I

b) Is it possible to trace a path along the arcs of figure II which passes through each juncture point once and only once?

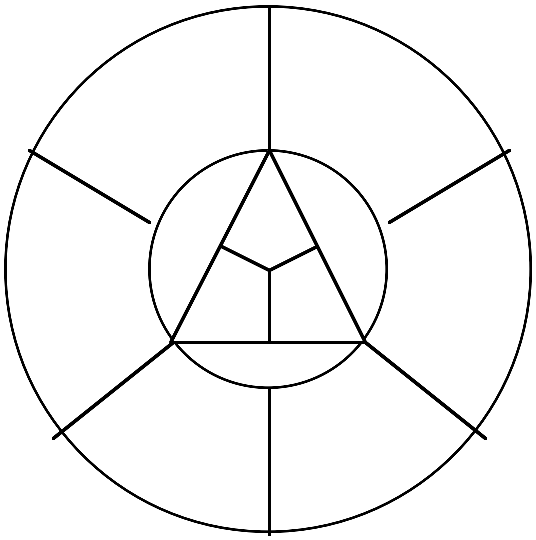

Figure II

II. Induction & Strong Induction

0. Prove: $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$

1. Prove the Binomial Theorem: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$.

2. If V, E, F, are respectively the number of vertices, edges, and faces of a connected planar map, then: V-E+F=2.

3. If a>0 and b>0, then $(n-1)a^n + b^n \geq na^{n-1}b$, where n is a positive integer, with equality iff a=b.

4. Prove:

a) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$

b) $2!4!\cdots(2n)! \geq [(n+1)!]^n$

4. The Euclidean plane is divided into regions by drawing a finite number of straight lines. Show that it is possible to color each of these regions either red or blue in such a way that no two adjacent regions have the same color.

5. Prove that $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$ is an integer for n=0,1,2,3....

6. For all $x \in [0, \pi]$ prove that $|\sin nx| \leq n \sin x$.

7.* Given a set of 51 integers between 1 and 100 (inclusive), show that at least one member of the set must divide another member of the set. (Hint: prove more generally that the same property will hold whenever n+1 integers are chosen from integers 1 to 2n (inclusive)).

8. Prove that the area of a simple lattice polygon (a polygon with lattice points as vertices whose sides do not cross) is given by $A_{lattice\ \ polygon} = I + \frac{B}{2} - 1$, where I and B denote respectively the number of interior and boundary points of the polygon.

III. Recursion

1. Suppose n rings, with different outside diameters, are slipped onto an upright peg, the largest on the bottom, to form a pyramid. Two upright pegs are placed sufficiently far apart. We wish to transfer all the rings, one at a time, to the second peg to form an identical pyramid. During the transfers, we are not permitted to place a larger ring on a smaller one. What is the smallest number of moves necessary to complete the transfer?

2. Find the sum of the first n squares:

a) $1^2 + 2^2 + 3^2 + ... + n^2 =$

b) $1^3 + 2^3 + 3^3 + ... + n^3 =$

c) $1^k + 2^k + 3^k + ... + n^k =$

IV. Pigeon Hole Principle

If kn+1, k≥ 1, objects are distributed among n boxes, then one of the boxes will contain at least k+1 objects.

-1. Prove that given any 5 points in the interior of an unit square at least one of the distances between any two points is less than $\frac{\sqrt{2}}{2}$.

0. 19 darts are thrown onto a dart board of the shape of a regular hexagon with the length of each side equal to 1 ft. Show that 2 darts are within $\frac{\sqrt{3}}{3}$ ft of one another.

1. Given a set of n+1 positive integers, none of which exceeds 2n, show that at least one member of the set must divide another member of the set.

2. Prove that there exist integers a, b, c, not all zero, where the absolute value of each one of them is less than a million, and $\left|a + b\sqrt{2} + c\sqrt{3}\right| < 10^{-11}$.

3. Given any set of 10 natural numbers between 1 and 99 inclusive, prove that there are two disjoint non empty subsets of the set with equal sums of their elements.

4. Paint each “unit” square of a 4 by 7 “chess” board either black or white. Prove that in any such coloring, the board must contain a rectangle whose distinct corner squares are of the same color.